UDC 538.4

CONTINUAL EQUATIONS OF ELECTRODYNAMICS OF CONDUCTING SUSPENSIONS MOVING IN A MAGNETIC FIELD

PMM Vol.41, № 1, 1977, pp.41-52

I. M. RUTKEVICH
(Moscow)

(Received August 20, 1975)

A continual model of the electrodynamics of a weakly concentrated suspension moving in a magnetic field is formulated. Averaged equations are derived with allowance for the presence of local parameter discontinuities at particle surfaces. The analysis of the current distribution in the neighborhood of a single particle is used for averaging Ohm's law. Expressions are established for macroscopic densities of electric power and Joule dissipation. Properties of continual equations for high and low Reynolds numbers are considered.

The problem of constructing continual electrodynamic models arises in connection with the formulation of equations of mechanics of suspensions considered as media with internal degrees of freedom [1] or having a micro-structure [2]. In such construction the passage to equations of the field averaged over physically small volumes with a large number of dispersed particles is presupposed.

A closed macrocontinual description becomes possible if the averaged material equations, in particular Ohm's law, are represented in the form of relationships between quantities that appear directly in the averaged laws of conservation. In stationary isotropic media the averaged Ohm's law is of the form $\langle \mathbf{j} \rangle = \sigma_e \langle \mathbf{E} \rangle$, where σ_e is the effective conductance, which for diluted suspension of spherical particles was determined by Maxwell [3]. Works on the modelling of effective conductance in the case of finite concentration and various structures of inclusions are available (see, e. g., [4, 5]).

In the case of magnetohydrodynamic interactions the averaged Ohm's law must be formulated with allowance for perturbations introduced by the motion of dispersed particles in the velocity field and, consequently, in the electric current density. Hence it is not possible to pass from the local Ohm's law to the macroscopic by a formal replacement of electrical conductivity by its effective value and the remaining quantities by their mean values. The relative motion of phases results in the appearance of "slip current" which may be considered as an internal degree of freedom.

1. The averaging of Maxwell equations. Let us consider a suspension of macro-particles with conductivity σ_s in a fluid or plasma with conductivity σ_f. Subscripts s and f relate to particles and fluid, respectively. The order of the minimum macro-volume in which the approximation of a continuous medium is valid is l^3. The quantity l, the characteristic dimension of particles a, and the characteristic length L of averaged parameter variation satisfy the inequality $a \ll l \ll L$.

The system is assumed to be nonpolarizable, nonferromagnetic, and quasi-neutral, hence the usual magnetohydrodynamic equations are applicable to both phases [6]. We assume that the condition

$$\min(\sigma_s, \sigma_f) = \sigma_0 \gg \varepsilon_0 \omega \tag{1.1}$$

is satisfied. In it ω is the characteristic frequency variation of local parameters and ε_0 is an electrical constant. (SI units are used throughout). In the absence of unstable external effects $\omega \sim V_s / d$, where V_s is the characteristic velocity of particles and d the mean distance between these. Inequality (1.1) makes it possible to neglect the slip current and assume the electric field to be quasi-stationary. The Maxwell equations for microfields may now be written as

$$\operatorname{rot} \mathbf{E} = -\partial \mathbf{B} / \partial t, \quad \operatorname{rot} \mathbf{B} = \mu_0 \mathbf{j} \tag{1.2}$$
$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{div} \mathbf{E} = \rho_e / \varepsilon_0$$

The term "microfield" is meant here to define the field of elementary charge carriers averaged on a scale that is considerably smaller than the dimension of dispersed particles, i.e. a field which in classical electrodynamics is considered as macroscopic. The result of averaging the microfield over volume D_l is a macrofield.

Since at the surface of particles electrical conductivity is discontinuous, the macrofield is also discontinuous. At these surfaces the conditions

$$\begin{aligned} &\mathbf{n}\{\mathbf{B}\} = 0, \quad \mathbf{n} \times \{\mathbf{B}\} = \mu_0 \mathbf{i}, \quad \mathbf{n}\{\mathbf{E}\} = \vartheta / \varepsilon_0 \\ &\mathbf{n} \times \{\mathbf{E}\} = -\mu_0 (\mathbf{n}\mathbf{V})\, \mathbf{n} \times \mathbf{i} + \mathbf{n} \times \nabla_\tau (q \mathbf{n} \mathbf{j}) \\ &\mathbf{n}\{\mathbf{V}\} = 0, \quad \mathbf{n}\{\mathbf{j}\} = -\nabla_\tau \cdot \mathbf{i} \end{aligned} \tag{1.3}$$

must be satisfied [6, 7]. In these formulas $\mathbf{n}$ is the external normal to the surfaces, $\mathbf{i}$ and ϑ are the densities of electric surface currents and charges, respectively, q is the contact resistance, and ∇_τ is the Hamilton surface operator. In the right-hand side of the fourth of Eqs. (1.3) only one term is nonzero, since the conditions of formation of current and electric double layers are usually incompatible.

Below we introduce the conventional operation of averaging over a physically small macrovolume D_l

$$\langle g \rangle = D_l^{-1} \int_{D_l} g(\mathbf{X} + \mathbf{x}')\, dD_l \tag{1.4}$$

where $\mathbf{X}$ is the radius vector of the center of mass of volume D_l. Vector $\mathbf{X} + \mathbf{x}'$ passes through all points of D_l, hence integration is carried out with respect to the variable $\mathbf{x}'$. Since function $\langle g \rangle$ is continuous at discontinuity surfaces of g, (1.4) is a smoothing operator. The space intervals $|d\mathbf{X}|$ and $|d\mathbf{x}|$, in which it is possible to consider the respective variations of functions $\langle g \rangle$ and g as small, satisfy the condition $|d\mathbf{X}| \gg a \gg |d\mathbf{x}|$. Because of this $\mathbf{X}$ may be taken as a macrocontinual radius vector. The invariant differential operators that are applied below to macroscopic functions presuppose differentiation with respect to X_i.

Permutability of differentiation and averaging operations applies to continuous fields. This property is used in the derivation of Maxwell equations from the equations of macroscopic electrodynamics [8]. In a suspension the fields may become discontinuous at phase interface boundaries, and this must be taken into consideration when averaging Eqs. (1.2). For function g which has discontinuities at the moving surfaces S_k that lie inside D_l, we can establish the following relationships:

$$\frac{\partial \langle g \rangle}{\partial t} = \left\langle \frac{\partial g}{\partial t} \right\rangle - \frac{1}{D_l} \sum_k \int_{S_k} V_n \{g\}\, dS_k, \quad V_n = \mathbf{n}\mathbf{V}_s = \mathbf{n}\mathbf{V}_f \tag{1.5}$$

$$\frac{\partial \langle g \rangle}{\partial X_i} = \left\langle \frac{\partial g}{\partial x_i} \right\rangle + \frac{1}{D_l} \sum_k \int_{S_k} n_i \{g\} dS_k, \quad \{g\} = g_f - g_s$$

where in this case S_k are surfaces of particles, n_i is the projection of $\mathbf{n}$ on the basis vector $\mathbf{e}_i$ and V_n is the normal velocity of the discontinuity propagation. Using (1.3)–(1.5), after averaging Eqs. (1.2), we obtain

$$\operatorname{rot} \langle \mathbf{E} \rangle = -\frac{\partial \langle \mathbf{B} \rangle}{\partial t} + \frac{1}{D_l} \sum_k \int_{S_k} \mathbf{n} \times \nabla_\tau (q\mathbf{n}\mathbf{j})\, dS_k \tag{1.6}$$

$$\operatorname{rot} \langle \mathbf{B} \rangle = \mu_0 \left(\langle \mathbf{j} \rangle + \frac{1}{D_l} \sum_k \int_{S_k} \mathbf{i}\, dS_k \right)$$

$$\operatorname{div} \langle \mathbf{B} \rangle = 0, \quad \operatorname{div} \langle \mathbf{E} \rangle = \varepsilon_0^{-1} \left(\langle \rho_e \rangle + \frac{1}{D_l} \sum_k \int_{S_k} \vartheta\, dS_k \right)$$

The surface integrals in (1.6) determine the contribution of micro-discontinuities to the generation of vortices and sources of the macroscopic field. The form of Eqs. (1.6) is independent of the volume concentration and shape of moving inclusions. Effect of contact resistance and the case of infinite electrical conductivity which require a special analysis are not considered here. Setting in (1.6) $q = 0$ and $\mathbf{i} = 0$, we obtain the system

$$\operatorname{rot} \langle \mathbf{E} \rangle = -\partial \langle \mathbf{B} \rangle / \partial t, \quad \operatorname{rot} \langle \mathbf{B} \rangle = \mu_0 \langle \mathbf{j} \rangle, \quad \operatorname{div} \langle \mathbf{B} \rangle = 0 \tag{1.7}$$

The last of Eqs. (1.6) is used for the determination of density of sources of the macroscopic field $\langle \mathbf{E} \rangle$, after the independent determination of the latter.

In what follows we consider a monodisperse suspension of undeformable spherical particles of radius a at low volume concentration

$$c = {}^4/_3 \pi a^3 n \ll 1 \tag{1.8}$$

where c and n are, respectively, the volume and numerical concentrations of particles. The microscopic current density is defined by the Ohm's law in its isotropic form

$$\mathbf{j} = \sigma (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \tag{1.9}$$

If the result of averaging the right-hand side of (1.9) reduces to a known function of $\langle \mathbf{E} \rangle$ and $\langle \mathbf{B} \rangle$ of mean velocities and conductivities of phases, it is possible to eliminate $\langle \mathbf{j} \rangle$ and $\langle \mathbf{E} \rangle$ from system (1.7) and obtain the equation of induction which relates vector $\langle \mathbf{B} \rangle$ to hydrodynamic characteristics. Lorentz volume forces are represented in the averaged equations of phase motion, and the equations of energy contain terms which define energy exchanges between each phase and the electromagnetic field. To determine the electric current contribution to the entropy generation of phases, it is necessary to know the average phase density of Joule dissipation. There arises the problem of representing these quantities in terms of $\langle \mathbf{j} \rangle$, $\langle \mathbf{E} \rangle$ and $\langle \mathbf{B} \rangle$ and of hydrodynamic properties. In the case of a weakly concentrated suspension the sought relationships are obtained by analyzing the problem of the electric field perturbed by particle motion in a boundless field.

The proposed course makes it possible to obtain equations for calculating averaged currents and fields using the specified mean velocities of components and particle concentration. Derivation of the complete system of magnetohydrodynamic equations is outside the scope of this article. It presupposes the determination of relations between the macrostress tensor and other dynamic characteristics of phases, and electromagnetic quantities. The force of interaction between phases, whish are determined by mechanical macrostresses at particle surfaces, depend in a conducting medium on the magnetic field. According to the investigations of the problem of flow past bodies in [9 — 11] the effect of a magnetic field on the hydrodynamic resistance is comparable to the action of the Lorentz force, and in some cases may even exceed it.

The averaging over phases is carried out in accordance with the rule

$$\langle g\rangle_\alpha = D_\alpha^{-1}\int_{D_\alpha} g(\mathbf{X}+\mathbf{x}')\,dD_\alpha \quad (\alpha = s, f)$$

where D_s and D_f are, respectively, the volumes occupied by particles and the fluid, and $D_s + D_f = D_l$. Below we make use of the relationship

$$\langle g\rangle = c\,\langle g\rangle_s + (1-c)\langle g\rangle_f \tag{1.10}$$

and of the formula for the mean product of two quantities

$$\langle g_1 g_2\rangle = \langle g_1\rangle\langle g_2\rangle + \langle \delta g_1 \delta g_2\rangle = \langle g_1\rangle\langle g_2\rangle + c\,\langle \delta g_1\delta g_2\rangle_s + (1-c)\,\langle \delta g_1\delta g_2\rangle_f, \quad \delta g_i = g_i - \langle g_i\rangle \tag{1.11}$$

Formulas (1.10) and (1.11) are also applicable to averages over elementary cells D_k. The latter represent regions occupied by the k-th particle and the surrounding fluid, as usually considered in suspension rheology [12]. The portions of volume occupied by the particle and the fluid are, respectively, c and $1-c$. The relation between mean values of cell parameters denoted below by quantities in brackets and the averages over volume D_l is of the form

$$n\langle g\rangle = \sum_k [g_k], \quad n\langle g\rangle_\alpha = \sum_k [g_k]_\alpha \quad (\alpha = s, f) \tag{1.12}$$
$$[g_k] = c\,[g_k]_s + (1-c)[g_k]_f$$

2. Electric current distribution in cells and averaging of Ohm's law. We assume that the definition of the microflow in a cell satisfies the inequalities

$$\sigma^\circ = \max(\sigma_s, \sigma_f) \ll (\mu_0\omega l^2)^{-1} \tag{2.1}$$
$$\mathrm{Re}_m^{(l)} = \sigma^\circ\mu_0 V^\circ l \ll 1, \quad V^\circ = \max(V_s, V_f)$$

The first of these inequalities makes it possible to neglect the derivative with respect to time in the first of Eqs. (1.2) and assume $\mathrm{rot}\,\mathbf{E} \approx 0$ within volume D_l, and the second of conditions (2.1) makes possible the assumption that $\mathbf{B} \approx \langle\mathbf{B}\rangle$ along a segment of length l.

Let us consider the current distribution when a spherical particle moves at velocity $\mathbf{V}_s$ in a stream whose velocity at infinity is $\mathbf{V}_\infty$ and the current density $\mathbf{j}_\infty$. For fairly small particles the Stewart number

$$\mathrm{St} = \sigma_f\,\langle B\rangle^2 a\,/\,(\rho_f W_\infty) \ll 1 \tag{2.2}$$

can be considered small.

(At very low velocity of flow W_∞ inequality (2.2) may be violated. In such case it is necessary to assume smallness of the square of the Hartmann number $\mathrm{Ha}^2 = \mathrm{St}\cdot\mathrm{Re} \ll 1$, where $\mathrm{Re} = W_\infty a\,/\,\nu_f$ is the usual Reynolds number). Owing to this it is possible, when determining the electric current in the first approximation by St , to consider the field of the fluid relative velocity W to be the same as that obtaining in the absence of magnetohydrodynamic interaction. The current density is determined by the Ohm's law

$$\mathbf{j}_s = \sigma_s \mathbf{E}', \quad \mathbf{j}_f = \sigma_f (\mathbf{E}' + \mathbf{W} \times \langle \mathbf{B} \rangle), \quad \mathbf{E}' = \mathbf{E} + \mathbf{V}_s \times \langle \mathbf{B} \rangle = -\nabla \Phi' \quad (2.3)$$

where $\mathbf{E}'$ is the electric field in a system moving with the particle and is determined by the solution of the boundary value problem for the potential Φ'

$$\begin{aligned} &\Delta\Phi_-' = 0 \quad (r < a), \quad \Delta\Phi_+' = \langle \mathbf{B} \rangle \operatorname{rot} \mathbf{W} \quad (r > a) \\ &\Phi_-' = \Phi_+', \quad \partial\Phi_-' / \partial r = \sigma_* (\partial\Phi_+' / \partial r - \mathbf{n} (\mathbf{W} \times \langle \mathbf{B} \rangle)) \quad (r = a) \\ &\nabla\Phi_+' \to -\mathbf{E}_\infty \quad (r \to \infty), \quad |\Phi_-'(0)| < \infty, \quad \sigma_* = \sigma_f / \sigma_s, \quad \mathbf{n} = \mathbf{r} / r \end{aligned} \quad (2.4)$$

where $\mathbf{r}$ is the radius vector drawn from the particle center, and Φ_-' and Φ_+' are the potentials inside and outside the sphere, respectively. It is assumed that the distribution $\mathbf{W}$ $(\mathbf{r})$ in the absence of a magnetic field is axisymmetric

$$\mathbf{W} = W_r (r, \theta)\mathbf{e}_r + W_\theta (r, \theta)\mathbf{e}_\theta \quad (2.5)$$

where the angle θ is read from the direction of $\mathbf{W}_\infty$. We assume the field $\mathbf{W}$ to be incompressible, which is admissible for $M^2 = W_\infty^2 / C_f^2 \ll 1$, where C_f is the speed of sound in the fluid

$$\operatorname{div} \mathbf{W} = 0 \quad (2.6)$$

The current distribution inside the particle is independent of the selected velocity field when the latter satisfies conditions (2.5) and (2.6); furthermore

$$\mathbf{n}\mathbf{W} = 0 \quad (r = a), \quad \mathbf{W} = \mathbf{W}_\infty + O(r^{-1}) \quad (r \to \infty) \quad (2.7)$$

To prove this statement it is sufficient to ascertain that the difference between the two solutions of (2.4) which correspond to velocity fields $\mathbf{W}_1$ and $\mathbf{W}_2$ is constant inside the sphere. Denoting that difference by χ, we obtain the following problem

$$\begin{aligned} &\Delta\chi_- = 0 \ (r < a), \ \Delta\chi_+ = \langle \mathbf{B} \rangle \operatorname{rot} \mathbf{u} \ (r > a) \\ &\chi_- = \chi_+, \ \partial\chi_- / \partial r = \sigma_* [\partial\chi_+ / \partial r - \mathbf{n} (\mathbf{u} \times \langle \mathbf{B} \rangle)] \ (r = a) \\ &\nabla\chi_+ \underset{r\to\infty}{=} O(r^{-1}), \quad \mathbf{u} = \mathbf{W}_2 - \mathbf{W}_1 \end{aligned} \quad (2.8)$$

It follows from (2.5) and (2.6) that $\mathbf{u}$ can be represented as

$$\mathbf{u} = \operatorname{rot} \Psi, \ \mathbf{\Psi} = \Psi (r, \theta)\mathbf{e}_\varphi \quad (2.9)$$

In conformity with the impermeability requirement Ψ vanishes when $r = a$. Let us consider the auxiliary external problem

$$\begin{aligned} &\Delta\chi_+^* = \langle \mathbf{B} \rangle \operatorname{rot} \mathbf{u} \quad (r > a) \\ &\partial\chi_+^* / \partial r = \mathbf{n} (\mathbf{u} \times \langle \mathbf{B} \rangle) \quad (r = a), \quad \nabla\chi_+^* \underset{r\to\infty}{=} O(r^{-1}) \end{aligned} \quad (2.10)$$

Using the relationships

$$\langle \mathbf{B} \rangle \operatorname{rot} \mathbf{u} = -\Delta (\langle \mathbf{B} \rangle \mathbf{\Psi})$$

$$\mathbf{n} (\mathbf{u} \times \langle \mathbf{B} \rangle) = -\frac{\partial}{\partial n} (\langle \mathbf{B} \rangle \mathbf{\Psi}) + \mathbf{n} (\langle \mathbf{B} \rangle \nabla) \mathbf{\Psi}$$

we conclude that problem (2.10) is equivalent to the following:

$$\Delta\Lambda = O \quad (r > a), \quad \partial\Lambda / \partial r = \mathbf{n} (\langle \mathbf{B} \rangle \nabla) \mathbf{\Psi} \quad (r = a) \quad (2.11)$$

$$\nabla \Lambda_{r\to\infty} = O(r^{-1}), \quad \Lambda = \chi_+^* + \langle \mathbf{B} \rangle \Psi$$

Since the identity

$$\mathbf{n}(\langle \mathbf{B} \rangle \nabla)\Psi = (\langle \mathbf{B} \rangle \nabla)(\mathbf{n}\Psi) - \Psi(\langle \mathbf{B} \rangle \nabla)\mathbf{n}$$

is valid and, owing to axial symmetry, $\mathbf{n}\Psi \equiv 0$ throughout the flow and at the sphere surface $\Psi = 0$, Neumann's condition in problem (2.11) vanishes, hence that problem has the solution $\Lambda \equiv C = \text{const}$. It is now evident that the pair of functions

$$\chi_- = C, \ \chi_+ = \chi_+^* = -\langle \mathbf{B} \rangle \Psi + C \tag{2.12}$$

satisfies all conditions (2.8). Since the solution of problem (2.8) is unique within the additive constant, (2.12) represents the sought solution.

Formula (2.7) makes it possible to calculate the current density in a particle (but not in the fluid) using the simplest field $\mathbf{W}$ which corresponds to the inviscid potential flow

$$\mathbf{W} = \mathbf{W}_1 = \nabla \{(1 + {}^1/{}_2 a^3 r^{-3}) \mathbf{W}_\infty \mathbf{r}\}$$

The solution for each field was obtained in [10] for the particular case of $\mathbf{j}_\infty = 0$. It is possible to extend it to the case of $\mathbf{j}_\infty \neq 0$ and, also, to take into account the vortex component of field $\mathbf{W}$ using formula (2.12). The corresponding formulas for the current and the electric field in the laboratory system of coordinates are of the form

$$\mathbf{j}_s = \frac{3\sigma_f}{1+2\sigma_*}\left(\mathbf{E}_\infty + \mathbf{V}_\infty \times \langle \mathbf{B} \rangle - \frac{1}{2}\mathbf{W}_\infty \times \langle \mathbf{B} \rangle\right) \tag{2.13}$$

$$\mathbf{j}_f = \sigma_f\Big(\mathbf{E}_\infty + \mathbf{V}_\infty \times \langle \mathbf{B} \rangle + \frac{1}{2}\nabla\left(\frac{a^3}{r^3}\mathbf{W}_\infty \mathbf{r}\right) \times \langle \mathbf{B} \rangle + \nabla\left\{\frac{a^3}{r^3}(\mathbf{E}_s - \mathbf{E}_\infty)\mathbf{r}\right\} + (\langle \mathbf{B} \rangle \nabla)\Psi\Big)$$

$$\mathbf{E}_s = (1 + 2\sigma_*)^{-1}(3\sigma_* \mathbf{E}_\infty - (1 - \sigma_*)\mathbf{V}_\infty \times \langle \mathbf{B} \rangle + (1 + {}^1/{}_2\sigma_*)\mathbf{W}_\infty \times \langle \mathbf{B} \rangle)$$

$$\mathbf{E}_f = \mathbf{E}_\infty + \nabla\left(\frac{a^3}{r^3}(\mathbf{E}_s - \mathbf{E}_\infty)\mathbf{r} + \langle \mathbf{B} \rangle \Psi\right)$$

In a weakly concentrated suspension the distribution (2.13) may be identified with the microvariable distribution inside elementary cells, whose averages over a cell are calculated by formula

$$[g] = D_R^{-1} \int_{D_R} g \, dD_R \tag{2.14}$$

where D_R denotes a sphere of radius $R = ac^{-1/3}$ with a center common with the particle. We shall use the following relationships:

$$[\mathbf{E}] = -D_R^{-1}\int_{D_R} \nabla\Phi \, dD_R = -D_R^{-1}\int_{S_R} \Phi \mathbf{n} \, dS_R \tag{2.15}$$

$$[\mathbf{j}] = D_R^{-1}\int_{D_R} \operatorname{div} \mathbf{T} \, dD_R = D_R^{-1}\int_{S_R} (\mathbf{n}\mathbf{j})'\mathbf{r} \, dS_R$$

where Φ is the electrical potential and $\mathbf{T}$ is the tensor with components $T_{ik} = r_i j_k$. The Gauss formula in (2.15) is used with allowance for the continuity of Φ and $\mathbf{nj}$ on the inner sphere S_a. It follows from (2.13) and (2.14) that on sphere S_R

$$\Phi = \Phi_f = -\{(1-c)\mathbf{E}_\infty + c\mathbf{E}_s\}\mathbf{r} - \langle \mathbf{B} \rangle \Psi \tag{2.16}$$

$$\mathbf{nj} = \mathbf{nj}_f = \mathbf{n}\{(1-c)\mathbf{j}_\infty + c\mathbf{j}_s + \sigma_f(\langle \mathbf{B} \rangle \nabla)\Psi\}$$

The substitution of (2.16) into (2.15) yields

$$[\mathbf{E}] = (1-c)\mathbf{E}_\infty + c\mathbf{E}_s + [\mathbf{E}_\Psi], \quad [\mathbf{j}] = (1-c)\mathbf{j}_\infty + c\mathbf{j}_s + [\mathbf{j}_\Psi] \quad (2.17)$$
$$[\mathbf{E}_\Psi] = -{}^1/_2[\operatorname{rot}\mathbf{\Psi}]\times\langle\mathbf{B}\rangle = \beta\mathbf{W}_\infty\times\langle\mathbf{B}\rangle, \quad [\mathbf{j}_\Psi] = -\sigma_f[\mathbf{E}_\Psi]$$

The representations of mean characteristics in the form of streams (2.15) reflect the combined effects of perturbations induced by particles. Parameter β generally depends on the Reynolds number. For an inviscid potential flow $\beta = 0$, while for the Stokes mode we have

$$\Psi = -\frac{3}{4}W_\infty a\left(1-\frac{a^2}{r^2}\right)\sin\theta, \quad \beta = \frac{3}{4}c^{1/3}(1-c^{2/3}) \quad (2.18)$$

Taking into account the homogeneity of quantities $\mathbf{E}_s$ and $\mathbf{j}_s$ inside a particle from (2.17) and the second of formulas (1.12), we obtain

$$[\mathbf{E}]_s = \mathbf{E}_s, \quad [\mathbf{j}]_s = \mathbf{j}_s, \quad [\mathbf{E}]_f = \mathbf{E}_\infty + \frac{[\mathbf{E}_\Psi]}{1-c}, \quad [\mathbf{j}]_f = \mathbf{j}_\infty + \frac{[\mathbf{j}_\Psi]}{1-c} \quad (2.19)$$

Using formulas (2.12), (2.13) and (2.19) and relationships

$$[\mathbf{V}]_s = \mathbf{V}_s, \quad [\mathbf{V}]_f = \mathbf{V}_\infty - 2\beta(1-c)^{-1}(\mathbf{V}_\infty - \mathbf{V}_s)$$

we establish the macroscopic Ohm's law

$$\langle\mathbf{j}\rangle = \sigma_e(\langle\mathbf{E}\rangle + \mathbf{U}_e\times\langle\mathbf{B}\rangle) \quad (2.20)$$
$$\sigma_e = \sigma_f\frac{1+2\varkappa c}{1-\varkappa c}, \quad \varkappa = \frac{1-\sigma_*}{1+2\sigma_*}, \quad \sigma_* = \frac{\sigma_f}{\sigma_s}$$
$$\mathbf{U}_e = \langle\mathbf{V}\rangle - \frac{3}{2}\frac{c(1-c)}{1+2\varkappa c}\varkappa\mathbf{W}_*, \quad \mathbf{W}_* = \langle\mathbf{V}\rangle_f - \langle\mathbf{V}\rangle_s$$

The quantity σ_e is the same as the effective conductivity of a suspension, as defined by Maxwell [3]. Vector $\mathbf{U}_e\times\langle\mathbf{B}\rangle$ determines the effective electromotive force generated by the motion of a two-phase medium in a magnetic field. When the main slip velocity $\mathbf{W}_*$ is nonzero, the equality $\mathbf{U}_e = \langle\mathbf{V}\rangle$ is satisfied if the conductivities of phases are the same. Note that parameter β, which depends on the detailed pattern of flow around particles, does not implicitly appear in the averaged Ohm's law. Formulas for the mean phase current densities are of the form

$$\langle\mathbf{j}\rangle_s = \frac{3}{(1+2\sigma_*)(1+2\varkappa c)}\{\langle\mathbf{j}\rangle - (1-c)\langle\mathbf{j}_*\rangle\} \quad (2.21)$$
$$\langle\mathbf{j}\rangle_f = (1+2\varkappa c)^{-1}\left\{\langle\mathbf{j}\rangle + \frac{3c}{1+2\sigma_*}\langle\mathbf{j}_*\rangle\right\}, \quad \langle\mathbf{j}_*\rangle = {}^1/_2\sigma_f\mathbf{W}_*\times\langle\mathbf{B}\rangle$$

This shows that when the mean current $\langle\mathbf{j}\rangle$ is passed through a two-speed suspension, the mean currents in phases cannot be determined by the conductivity of these and by the quantity c, if the "slip current" $\langle\mathbf{j}_*\rangle$ is not specified.

The derivation of Ohm's law described above is based on current distribution in cells with translational movement of particles. Besides distributions (2.13) components of microfields $\mathbf{E}_\omega$ and current $\mathbf{j}_\omega$ are possible. These are due to the rotation of particles at angular velocity $\boldsymbol{\omega}_s$ and surrounding fluid having an undisturbed angular velocity $\boldsymbol{\omega}_f$ for $r\to\infty$. The electrical potential Φ_ω induced by the slow relative rotation of the sphere is determined by the solution of problem

$$\Delta\Phi_\omega^- = 2\langle\mathbf{B}\rangle\boldsymbol{\omega}_s \quad (r<a), \quad \Delta\Phi_\omega^+ = \langle\mathbf{B}\rangle\operatorname{rot}\mathbf{V}_\omega \quad (r>a) \quad (2.22)$$
$$\Phi_\omega^- = \Phi_\omega^+, \quad \partial\Phi_\omega^-/\partial r + a\{(\mathbf{n}\langle\mathbf{B}\rangle)(\mathbf{n}\boldsymbol{\omega}_s) - \langle\mathbf{B}\rangle\boldsymbol{\omega}_s\} =$$
$$\sigma_*\{\partial\Phi_\omega^+/\partial r - \mathbf{n}(\mathbf{V}_\omega\times\langle\mathbf{B}\rangle)\} \quad (r=a)$$

$$\nabla\Phi_{\omega}^{+}\big|_{r\to\infty}=\frac{2}{3}(\langle\mathbf{B}\rangle\,\boldsymbol{\omega}_f)\,\mathbf{r}+o(1),\quad \mathbf{V}_{\omega}=\boldsymbol{\omega}_f\times\mathbf{r}+\frac{a^3}{r^3}(\boldsymbol{\omega}_s-\boldsymbol{\omega}_f)\times\mathbf{r}$$

where $\mathbf{V}_{\omega}$ is the inertia-free approximation of the velocity field of a viscous fluid in the presence of rotation of the sphere and fluid stream. The condition for $r\to\infty$ is formulated with allowance for $\operatorname{div}\mathbf{j}_{\omega}=0$. The solution of problem (2.22) is of the form

$$\Phi_{\omega}^{-}=\frac{1}{3}\langle\mathbf{B}\rangle\{\boldsymbol{\omega}_s(r^2-a^2)+2(\boldsymbol{\omega}_f-\boldsymbol{\omega}_s)a^2\}+\lambda_s Z_s-\lambda_f Z_f \tag{2.23}$$

$$\Phi_{\omega}^{+}=\frac{1}{3}\langle\mathbf{B}\rangle\left\{\boldsymbol{\omega}_f(r^2-a^2)+2(\boldsymbol{\omega}_f-\boldsymbol{\omega}_s)\frac{a^3}{r}\right\}+$$
$$+\left\{\frac{a^3}{2r^3}+\left(\lambda_s-\frac{1}{2}\right)\frac{a^5}{r^5}\right\}Z_s-$$
$$-\left\{\frac{a^3}{2r^3}+\left(\lambda_f-\frac{1}{2}\right)\frac{a^5}{r^5}\right\}Z_f,\quad Z_{\alpha}=\frac{1}{3}r^2\boldsymbol{\omega}_{\alpha}\langle\mathbf{B}\rangle-(\mathbf{r}\boldsymbol{\omega}_{\alpha})(\mathbf{r}\langle\mathbf{B}\rangle)$$
$$\lambda_s=(2+3\sigma_*)^{-1},\quad \lambda_f=(2+3\sigma_*^{-1})^{-1}$$

where the unimportant additive constant in the formula for Φ_{ω} is omitted. With the use of formulas (2.15) and (2.23) it is possible to ascertain that $[\mathbf{E}_{\omega}]=-[\nabla\Phi_{\omega}]=0$ and $[\mathbf{j}_{\omega}]=0$. This result remains valid in the case of quasi-solid rotation of a perfect fluid when $\mathbf{V}_{\omega}\equiv\boldsymbol{\omega}_f\times\mathbf{r}$. The induced electric field with a sphere rotating in an irrotation stream of perfect fluid was investigated in [10]. When the inequalities

$$\mathrm{Re}=W_{\infty}a/\nu_f\ll 1,\quad \mathrm{Re}_{\omega}=|\boldsymbol{\omega}_f-\boldsymbol{\omega}_s|\,a^2/\nu_f\ll 1$$

are satisfied, the perturbation of the velocity field in the fluid is a superposition of fields that correspond to the translational and rotational motions of a particle. Because of this, $\mathbf{E}_{\omega}$ and $\mathbf{j}_{\omega}$ appear in local distributions as a sum, thus leaving the mean values $[\mathbf{E}]$ and $[\mathbf{j}]$, and the form of Ohm's macroscopic law unchanged.

3. Macroscopic densities of power and dissipation. The solution derived in Sect. 2 may be used for the determination of densities of electric power - $[\mathbf{jE}]$ and dissipation $[\mathbf{j}^2/\sigma]$ averaged over cells, and for the subsequent calculation of the macrodensity of these quantities. In conformity with (1.11) we have

$$[\mathbf{jE}]=[\mathbf{j}][\mathbf{E}]+[\delta\mathbf{j}\delta\mathbf{E}] \tag{3.1}$$

The correlation term may be represented in the flux form

$$[\delta\mathbf{j}\delta\mathbf{E}]=-D_R^{-1}\int_{D_R}(\mathbf{j}-[\mathbf{j}])\nabla(\Phi+[\mathbf{E}]\mathbf{r})\,dD_R= \tag{3.2}$$
$$-D_R^{-1}\int_{S_R}(\Phi+[\mathbf{E}]\mathbf{r})\,\mathbf{n}(\mathbf{j}-[\mathbf{j}])\,dS_R$$

In the case of distribution (2.13) which relates to the transtational motion of the particle the surface integral in (3.2) can be reduced with allowance for (2.13) and (2.16) to the form

$$[\delta\mathbf{j}\delta\mathbf{E}]=-\sigma_f(D_R R)^{-1}\int_{S_R}\left\{\langle\mathbf{B}\rangle\Psi(r,\theta)-\right. \tag{3.3}$$
$$\left.\frac{3}{4}\sin\theta\int_0^{\pi}\langle\mathbf{B}\rangle\Psi(r,\theta')\sin^2\theta'\,d\theta'\right\}^2 dS_R$$

The nonpositivity of the correlation term means that the elementary cell generates electrical power in addition to $-[\mathbf{j}][\mathbf{E}]$. Such generation is admittedly absent if the flow past the particle is potential or is parallel to the magnetic field. If the representation $\Psi = f(r)\sin\theta$ is valid, which is the case with Stokes flow modes (see formula (2.18)), and also $[\delta\mathbf{j}\delta\mathbf{E}] = 0$.

If in the case of low Reynolds numbers the rotational effects are taken into account, it is necessary to add to the correlation term (3.3) the term $[\mathbf{j}_\omega\mathbf{E}_\omega]$. Cross terms of the form $[\mathbf{j}_t\mathbf{E}_\omega]$ and $[\mathbf{j}_\omega\mathbf{E}_t]$, where $\mathbf{j}_t$ and $\mathbf{E}_t$ are distributions, determined by (2.13) which relate to translational motion, prove to be zero. For the quantity $[\mathbf{j}_\omega\mathbf{E}_\omega]$ obtained from the solution (2.13) we have

$$[\mathbf{j}_\omega\mathbf{E}_\omega] = {}^1/_{30}\sigma_f R^2(\omega_f - \omega_s)\{3\langle B\rangle^2\omega_f + (\langle\mathbf{B}\rangle\omega_f)\langle\mathbf{B}\rangle\}c(1 + O(c^{2/3})) \qquad (3.4)$$

where $R = ac^{-1/3}$ is the radius of the cell. If $|\omega_f - \omega_s| \lesssim \omega_f$, the ratio of the right-hand side of (3.4) to $[\mathbf{j}][\mathbf{E}]$ is of order $cR^2\omega_f^2/(10 V_\infty^2)$. When the inequality $R^2/L^2 \ll 10$, where L is the characteristic length of mean velocity variation, the above ratio can be neglected within the scope of the theory that is linear with respect to particle concentration.

Thus for low Reynolds numbers it is possible to assume

$$[\mathbf{jE}] = [\mathbf{j}][\mathbf{E}] \qquad (3.5)$$

Summing formulas (3.5) over the cells and considering $\mathbf{V}_\infty$ and $\mathbf{E}_\infty$ as constant in D_l, with the use of (1.12) and (2.17) we obtain

$$\langle\mathbf{jE}\rangle = \langle\mathbf{j}\rangle\langle\mathbf{E}\rangle - \left\{\frac{(1+\sigma_*/2)c}{1+2\sigma_*} - \beta\right\} \times \left\{\frac{3c/2}{1+2\sigma_*} - \beta\right\}\sigma_f n^{-1}\sum_k \{(\langle\mathbf{V}\rangle_s - \mathbf{V}_k)\times\langle\mathbf{B}\rangle\}^2 \qquad (3.6)$$

where $\mathbf{V}_k$ is the velocity of the k-th particle center of mass. The correlation correction to $\langle\mathbf{j}\rangle\langle\mathbf{E}\rangle$ defines the electric power generation in the macrovolume owing to the particle distribution by velocities. At low Reynolds numbers that correction is proportional to $c^{2/3}$, but when the distribution dispersion ξ is fairly small ($\xi \ll \langle V\rangle_s^2 c^{1/3}$) the ratio of the correlation term to $\langle\mathbf{j}\rangle\langle\mathbf{E}\rangle$ is considerably smaller than c. Then in the linear approximation by c we have

$$\langle\mathbf{jE}\rangle = \langle\mathbf{j}\rangle\langle\mathbf{E}\rangle \qquad (3.7)$$

Formula (3.7) is even more valid for the model of irrotational flow past particles, since for $\beta = 0$ the correlation term is of order c^2.

The Joule dissipation density averaged over cells substantially depends on function Ψ. For the Stokes mode of flow from (2.13) and (2.17) we obtain

$$\left[\frac{j^2}{\sigma}\right] = \frac{[\mathbf{j}]^2}{\sigma_e} + \frac{9}{40}\sigma_f\{7(\mathbf{W}_\infty\times\langle\mathbf{B}\rangle)^2 + (\mathbf{W}_\infty\langle\mathbf{B}\rangle)^2\}c^{2/3} + O(c) \qquad (3.8)$$

In conformity with (2.23) the allowance for rotational motion results in a dissipative term of the form

$$[j_\omega^2/\sigma] = {}^1/_5\sigma_f R^2\{\langle B\rangle^2\omega_f^2 - {}^1/_3(\langle\mathbf{B}\rangle\omega_f)^2\}(1 + O(c))$$

whose ratio to the first term of (3.8) is small in comparison with c when $R^2/L^2 \ll c$. In that case the effects of rotation and dissipation can be neglected.

For the potential flow past a particle we have

$$\left[\frac{j^2}{\sigma}\right] = \frac{[\mathbf{j}]^2}{\sigma_e} + \frac{c(1-c)}{\sigma_f(1+2\sigma_*+2(1-\sigma_*)c)}\{3(3+\sigma_*+(1-\sigma_*)c)[\mathbf{j}_*]^2 - \qquad (3.9)$$

$$2(1-\sigma_*)[\mathbf{j}_*][\mathbf{j}]\}, \quad [\mathbf{j}_*] = {}^1/_2\sigma_f \mathbf{W}_\infty \times \langle \mathbf{B} \rangle$$

We summate formulas (3.9) over the cells taking into account formulas (1.12) and (2.17) and neglecting the dissipation term of order c^2 similar to that in (3.6) and related to particle distribution by velocities. Then the macrodensity $\langle j^2 / \sigma \rangle$ in the case of irrotation flow in linear approximation by c is defined by

$$\left\langle \frac{j^2}{\sigma} \right\rangle = \frac{\langle \mathbf{j} \rangle^2}{\sigma_e} + \frac{c}{\sigma_f(1+2\sigma_*)} \{3(3+\sigma_*)\langle \mathbf{j}_* \rangle^2 - 2(1-\sigma_*)\langle \mathbf{j}_* \rangle \langle \mathbf{j} \rangle\} \quad (3.10)$$

Formula (3.10) implies that the averaged dissipation does not vanish when the current mean density is zero. This is explained by the thermal effects of microcurrents whose generation is unavoidable when the flow past particles is oblique to the magnetic field.

4. Continual equations at high and low magnetic Reynolds numbers. The elimination of quantities $\langle \mathbf{E} \rangle$ and $\langle \mathbf{j} \rangle$ from (2.20) and the first two of Eqs. (1.7) makes it possible to obtain for the induction an equation which defines the magnetic field in a two-phase medium

$$\frac{\partial \langle \mathbf{B} \rangle}{\partial t} = \operatorname{rot}(\mathbf{U}_e \times \langle \mathbf{B} \rangle) + \frac{1}{\mu_0 \sigma_e}(\nabla \ln \sigma_e \times \operatorname{rot}\langle \mathbf{B} \rangle + \Delta \langle \mathbf{B} \rangle) \quad (4.1$$

In the case of high magnetic Reynolds numbers $\mathrm{Re}_m^{(L)} = \mu_0 \sigma_e U_e L \gg 1$ the first term in the right-hand side of (4.1) is the principal one. If the diffusion terms are neglected, it is possible to obtain the analog of the condition of the magnetic field freezing-in in the case of a two-phase medium [6, 7]. However, unlike in the case of the homogeneous medium, the magnetic field does not move with the material volume but is attached to a fictituous continuum which moves at velocity $\langle \mathbf{V} \rangle + {}^3/_4 c\mathbf{W}_*$.

The effect of the gradient of macroparticle concentration on the propagation of magnetic perturbations is another peculiarity of suspensions with phases of constant conductivity. We transform the gradient in (4.1) restricting in the second of formulas (2.20) the approximation for σ_e to a linear one with respect to c

$$\nabla \ln \sigma_e \approx 3\varkappa \nabla c \quad (4.2)$$

Substituting (4.2) into (4.1) and assuming that the coefficients of the obtained equation are constant, for plane waves of the form

$$\langle \mathbf{B} \rangle = \mathbf{b} \exp\{i(\mathbf{KX} - \omega t)\}, \quad \operatorname{Im} \mathbf{K} = 0, \quad \mathbf{Kb} = 0, \quad \omega = \omega_r + i\omega_i$$

we obtain the following dispersion formulas:

$$\omega_r = (\mathbf{U}_e + 3\mu_0^{-1}\sigma_e^{-1}\varkappa\nabla c)\,\mathbf{K}, \quad \omega_i = -\mu_0^{-1}\sigma_e^{-1}K^2 \cdot \quad (4.3)$$

This shows that in the presence of a concentration gradient the rate of perturbation transfer is further increased in the direction of $\varkappa \nabla c$ due to the incease of effective conductivity.

Electric current distribution in channels at low Reynolds numbers are subject of detailed investigations in connection with various applications in magnetohydrodynamics (see, e. g. [13]). Similar steady state problems with $\mathrm{Re}_m^{(L)} \ll 1$ may be considered in the case of two-phase media, when $\langle \mathbf{B} \rangle = \mathbf{B}_0 + \mathbf{B}_i$, where $\mathbf{B}_0$ and $\mathbf{B}_i$ are, respectively, the external and the induced magnetic fields which can be neglected in Ohm's law (2.20). The distribution $\langle \mathbf{E} \rangle$ and $\langle \mathbf{j} \rangle$ are then determined by the solution of system

$$\operatorname{rot}\langle \mathbf{E} \rangle = 0, \quad \operatorname{div}\langle \mathbf{j} \rangle = 0, \quad \langle \mathbf{j} \rangle = \sigma_e(\langle \mathbf{E} \rangle + \mathbf{U}_e \times \mathbf{B}_0) \quad (4.4)$$

We resort to macrocontinual equations with the intention of using these for calculating the over-all electrical characteristics of two-phase systems. This is justified if it yields adequate results of integration of the true and averaged distributions over macroscopic manifolds. With the use of definition (1.4) it is possible to show that the integrals of micro- and macrodistributions taken over the finite volume $D \sim L^3$ coincide to within the term of order l / L. Since, strictly speaking, the averaging in (1.4) presupposes passing to limit $l / L \to 0$ with fixed ratio a / l [12], the over-all electric power N and dissipation Q can be calculated by formulas

$$N = -\int_D \langle \mathbf{jE} \rangle \, dD, \quad Q = \int_D \left\langle \frac{j^2}{\sigma} \right\rangle dD$$

On the basis of formula (3.7) we conclude that N can be determined by solving system (4.4). However the solutions of the latter are insufficient for determining Q, since macrodensity of dissipation is not determined by specifying vectors $\langle \mathbf{j} \rangle$ and $\mathbf{U}_e \times \mathbf{B}_0$.

Total current through the macrosurface and the difference of potentials between two points at a distance of order L can be determined in the steady state case with the use of integrals of averaged functions. This follows from the integral laws of conservation for microscopic quantities

$$\oint_\Sigma \mathbf{j}\boldsymbol{\nu} \, d\Sigma = 0, \quad \oint_\Gamma \mathbf{E}\boldsymbol{\tau} \, d\Gamma = 0 \tag{4.5}$$

where $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are unit vectors of the normal to surface Σ and tangent to curve Γ, respectively. Using (1.4) and (4.5) we obtain formulas

$$\int_F \langle \mathbf{j} \rangle \boldsymbol{\nu} \, dF = \left\{1 + O\left(\frac{l}{L}\right)\right\} \int_F \mathbf{j}\boldsymbol{\nu} \, dF \tag{4.6}$$

$$\int_C \langle \mathbf{E} \rangle \boldsymbol{\tau} \, dC = \left\{1 + O\left(\frac{l}{L}\right)\right\} \int_C \mathbf{E}\boldsymbol{\tau} \, dC$$

where F is an open surface $\sim L^2$ and C is a curve (segment) whose end points are at a distance $\sim L$ from each other.

If the microdifferential of length δC is of the order l_*, then in order to pass to macrodefinition it is necessary that $l \ll l_* \ll L$. Assuming that within a scale of order l_* the averaged functions are constant and applying (4.6) to the macroscopically small area δF and arc δC, we obtain

$$\langle \mathbf{j} \rangle \boldsymbol{\nu}_* \delta F = \left\{1 + O\left(\frac{l}{l_*}\right)\right\} \int_{\delta F} \mathbf{j}\boldsymbol{\nu} \, dF, \quad \langle \mathbf{E} \rangle \boldsymbol{\tau}_* \delta C = \left\{1 + O\left(\frac{l}{l_*}\right)\right\} \int_{\delta C} \mathbf{E}\boldsymbol{\tau} \, dC$$

where $\boldsymbol{\nu}_*$ and $\boldsymbol{\tau}_*$ are macroscopic vectors of the normal and tangent introduced in [12]. Neglecting terms of order l / l_* it is possible to conclude that the normal component of vector $\langle \mathbf{j} \rangle$ at the impermeable wall and the tangent component of $\langle \mathbf{E} \rangle$ at the perfect electrode are absent. This makes it possible to apply the conventional boundary value problem formulation to system (4.4). The use of Ohm's formula in the form (4.4) close to a wall is admissible, if in that region the condition of smallness of particle volume concentration is not violated.

REFERENCES

1. Sedov, L. I., Models of continuous media with internal degrees of freedom. PMM Vol. 32, № 5, 1968.
2. Batchelor, G. K., The stress system in a suspension of force-free particles. J. Fluid Mech., Vol. 41, pt. 3, 1970.
3. Maxwell, J. C., A Treatise on Electricity and Magnetism. Vols. 1-2, Oxford, Clarendon Press, 1873. Dover Publ. Inc., 1954.
4. Kerner, E. H., The electrical conductivity of composite media. Proc. Phys. Soc., Sect. B, Vol. 69, № 8, 1956.
5. Dykhne, A. M., The conductivity of two-dimensional two-phase system. ZhETF, Vol. 59, № 1, 1970.
6. Kulikovskii, A. G. and Liubimov, G. A., Magnetohydrodynamics. Moscow, Fizmatgiz, 1962.
7. Shercliff, J. A., A Course of Magnetohydrodynamics. Moscow, "Mir", 1967.
8. Tamm, I. E., Basis of the Theory of Electricity. Moscow, "Nauka", 1966.
9. Imai, I., On flows of conducting fluids past bodies. Rev. Mod. Phys., Vol. 32, № 4, 1960.
10. Reitz, J. R. and Foldy, L. L., The force on a sphere moving through a conducting fluid in the presence of a magnetic field. J. Fluid Mech., Vol. 11, pt. 1, 1961.
11. Gourdine, M. C., Magnetohydrodynamic flow construction with fundamental solutions. J. Fluid Mech., Vol. 10, pt. 3, 1961.
12. Brenner, H., Rheology of two-phase systems. Annual Rev. Fluid Mech., Vol. 2, Palo Alto, Annual Rev. Inc., 1970.
13. Vatazhin, A. B., Liubimov, G. A. and Regirer, S. A., Magnetohydrodynamic Flows in Channels. Moscow, "Nauka", 1970.

Translated by J. J. D.